9. Rovnice a nerovnice s neznámou pod odmocninou
(iracionální rovnice a nerovnice)

Ⓐ ROVNICE – při řešení využíváme

- DŮSLEDKOVÉ (NEEKVIVALENTNÍ) ÚPRAVY: umocňování obou stran rovnice (mohou přibýt kořeny) => ZKOUŠKA SOUČÁSTÍ ŘEŠENÍ
- EKVIVALENTNÍ ÚPRAVY: umocňování obou stran rovnice, pokud jsou obě strany ... nezáporné (≥ 0), nebo obě strany nekladné (≤ 0)
  - pokud určování podmínek zdlouhavé => lépe neekviv. + zkouška

- $\sqrt{a}$ je defin. pro $a \geq 0$ $\quad \sqrt{a} \geq 0 \quad \sqrt{a^2} = |a|$

Příklady

① Řešte v R

a) $\sqrt{2x+4} = 3 \quad /^2$

1. zp. – důsled. úpr. + zk.

$2x+4 = 9$

$2x = 5$

$x = \frac{5}{2}$

Zk.: $L = \sqrt{2\cdot\frac{5}{2}+4} = \sqrt{9} = 3$

$P = 3 \qquad L = P$

$K = \{\frac{5}{2}\}$

2. zp. – ekviv.

$\underbrace{\sqrt{2x+4}}_{\geq 0} = \underbrace{3}_{\geq 0} \quad /^2$ ekv.

[$2x+4 \geq 0$; $2x \geq -4$; $x \geq -2$ def. obor; obě strany ≥ 0: $\sqrt{2x+4} \geq 0$ pl., $3 \geq 0$ pl.; $D = \langle -2; +\infty)$]

$2x+4 = 9$

$2x = 5$

$x = \frac{5}{2} \in D \qquad K = \{\frac{5}{2}\}$

b) $\sqrt{x+7} = x-5 \quad /^2$

$x+7 = (x-5)^2$! umocň. celou stranu jako dvojčlen $(a \pm b)^2 = a^2 \pm 2ab + b^2$

$x+7 = x^2 - 10x + 25$

$0 = x^2 - 11x + 18$

$0 = (x-9)(x-2)$

$x_1 = 9 \in D$ ekv. $\qquad x_2 = 2 \notin D$ ekv.

! ekviv. $D = D' \cap$ podmínky na ekviv.

! NESTAČÍ POUZE URČIT def. obor D'

[$x+7 \geq 0 \ \wedge$ podm. na ekviv. $\sqrt{x+7} \geq 0$ pl. $\wedge \ x-5 \geq 0$; $x \geq -7$ (D'); $x \geq 5$; číselná osa: -7, 5; $D = \langle 5; +\infty)$]

neekv. (nemusíme D a podm., ale ZK

Zk.: $L(9) = \sqrt{16} = 4 \quad P(9) = 9-4 = 4 \quad L(9) = P(9)$

$L(2) = \sqrt{9} = 3 \quad P(2) = 2-5 = -3 \quad L(2) \neq P(2)$

$K = \{9\}$

c) $\sqrt{1+x} - \sqrt{4-x} = 1$ VÍCE ODMOCNIN => 1 OSAMOSTATNÍME a umocň. => postup opak.

1. zp. neekv.

$\sqrt{1+x} = (1 + \sqrt{4-x}) \quad /^2$

$1 + x = 1 + 2\sqrt{4-x} + 4 - x$

$2x - 4 = 2\sqrt{4-x} \quad /:2$

$(x-2) = \sqrt{4-x}$

$x^2 - 4x + 4 = 4 - x$

$x^2 - 3x = 0$

$x(x-3) = 0$

$x_1 = 0 \qquad x_2 = 3$

[pokud ekv. => podmínky složitější než zk. $x+1 \geq 0 \ \wedge \ \sqrt{1+x} - \sqrt{4-x} \geq 0$, protože $1 \geq 0$ (P strana); $x \geq -1$; D' JEN D' nestačí]

$L(0) = \sqrt{1} - \sqrt{4} = 1 - 2 = -1$

$P(0) = 1 \qquad L(0) \neq P(0)$

$L(3) = \sqrt{4} - \sqrt{1} = 2 - 1 = 1$

$P(3) = \sqrt{4} - \sqrt{1} = 2 - 1 = 1$

$K = \{3\}$

[pozn. - pokud neosamostatníme 1 odmocninu

2.zp. $\sqrt{1+x} - \sqrt{4-x} = 1 \quad /^2$

$1+x - 2\sqrt{1+x}\sqrt{4-x} + 4 - x = 1$

$4 - 2\sqrt{(1+x)(4-x)} = 0 \quad /:2$

$2 - \sqrt{(1+x)(4-x)} = 0 \quad \longrightarrow \quad 2 = \sqrt{(1+x)(4-x)} \quad /^2$

$4 - 2\sqrt{(1+x)(4-x)} + (1+x)(4-x) = 0$

odmocnina nemizí => osamostatnit

$4 = (1+x)(4-x)$

$4 = 4 - x + 4x - x^2$

$0 = -x^2 + 3x$

$x^2 - 3x = 0$

$x(x-3) = 0$

$x_1 = 0 \quad x_2 = 3 \quad \to$ ZK. nebo podm. + D

POROVNEJ S 1. ZP.

=> SLOŽITĚJŠÍ 2. ZP.]

d) $\sqrt{5-5y} - \sqrt{3y-11} = 0$

neekv. $\sqrt{5-5y} = \sqrt{3y-11}$

$5 - 5y = 3y - 11$

$16 = 8y$

$y = 2$

Zk. $L(2) = \sqrt{5-10} - \sqrt{3\cdot 2 - 11} = \sqrt{-5} - \sqrt{-5}$! nedef.

$P(2) = 0$

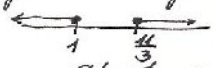
$\mathcal{K} = \emptyset$

ekv. $\mathcal{D}'$

[$5 - 5y \geq 0 \wedge 3y - 11 \geq 0$ a obě strany ≥ 0

$-5y \geq -5 \qquad 3y \geq 11$

$y \leq 1 \qquad y \geq \frac{11}{3}$



$\mathcal{D}' = \emptyset$ => nemusíme nic počítat - rce NEMÁ v $\mathcal{D} = \emptyset$ žádné řeš.]

e) $2\sqrt{x^2 - x + \frac{1}{4}} = (1 - 2x) \quad /^2$

$4(x^2 - x + \frac{1}{4}) = 1 - 4x + 4x^2$

$4x^2 - 4x + 1 = 1 - 4x + 4x^2$

$0x = 0$

$x \in R$ lib., ale jak zkoušku?

=> dopočítáme podmínky na ekv. úpravy

[ekv.

$L \geq 0$ vždy => musí být $P \geq 0$

$\mathcal{D}'$: $x^2 - x + \frac{1}{4} \geq 0 \quad \wedge \quad 1 - 2x \geq 0$

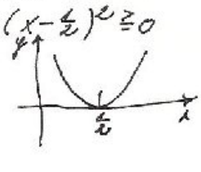
$4x^2 - 4x + 1 \geq 0 \qquad 2x \leq 1$

[nul. b. $x_{1,2} = \frac{4 \pm \sqrt{16-16}}{8}$ $\qquad x \leq \frac{1}{2}$

$x_{1,2} = \frac{1}{2}$]

$4(x - \frac{1}{2})(x - \frac{1}{2}) \geq 0 \quad \wedge \quad x \leq \frac{1}{2}$

$4(x - \frac{1}{2})^2 \geq 0 \qquad x \leq \frac{1}{2}$



$\mathcal{D} = (-\infty, \frac{1}{2})$]

$\mathcal{K} = \mathcal{D}$

$\mathcal{K} = (-\infty, \frac{1}{2}\rangle$

② Řešte v R

a) $\sqrt{2x + 1 + 2\sqrt{2x+3}} = 1 \quad /^2$ neekv.

$2x + 1 + 2\sqrt{2x+3} = 1 \quad /:2$ (osamos. √)

$\sqrt{2x+3} = -x \quad /^2$

$2x + 3 = x^2$

$0 = x^2 - 2x - 3$

$0 = (x-3)(x+1)$

$x_1 = 3 \qquad x_2 = -1$

[podm. na ekv. vedou k irac. nerci

$\sqrt{2x + 1 + 2\sqrt{2x+3}} \geq 0$ => neekv. + ZK]

$L(3) = \sqrt{6 + 1 + 2\sqrt{6+3}} = \sqrt{7+6} = \sqrt{13}$

$P(3) = 1 \qquad L(3) \neq P(3)$

$L(-1) = \sqrt{-2 + 1 + 2\sqrt{-2+3}} = \sqrt{-1+2} = 1$

$P(-1) = 1 \qquad L(-1) = P(-1)$

$\mathcal{K} = \{-1\}$

b) $\sqrt{x+25} - 4 = \frac{\sqrt{x^2+25x+4}}{\sqrt{25+x}+4}$

$(\sqrt{x+25}-4)(\sqrt{x+25}+4) = \sqrt{x^2+25x+4}$ $\quad (a-b)(a+b)$

$x+25-16 = \sqrt{x^2+25x+4}$ $\quad a^2-b^2$

$(x+9) = \sqrt{x^2+25x+4} \quad /^2$

$x^2+18x+81 = x^2+25x+4$

$77 = 7x \quad /:7$

$x = 11$

Zk.: $L = \sqrt{11+25} - 4 = 6-4 = 2$

$P = \frac{\sqrt{121+275+4}}{\sqrt{25+11}+4} = \frac{\sqrt{400}}{6+4} = \frac{20}{10} = 2$

$L = P \qquad \mathcal{K} = \{11\}$

c) DÚ $\frac{\sqrt{x+13}+2}{\sqrt{x+13}-4} = 7$

$\sqrt{x+13}+2 = 7(\sqrt{x+13}-4)$

$\sqrt{x+13}+2 = 7\sqrt{x+13}-28 \quad /-1\sqrt{x+13}+28$

$6\sqrt{x+13} = 30 \quad /:6$ [! JINAK PAK VELKÁ ČÍSLA => CHYBY]

$\sqrt{x+13} = 5$

$x+13 = 25 \quad /-13$

$x = 12$

Zk.: $L(12) = \frac{\sqrt{25}+2}{\sqrt{25}-4} = \frac{7}{1} = 7$

$P(12) = 7 \qquad L = P \qquad \mathcal{K} = \{12\}$

③ Řešte v R

a) DÚ $\sqrt{x^2-4x} = x-3$

$x^2-4x = x^2-6x+9$

$2x = 9$

$x = \frac{9}{2}$

Zk.: $L = \sqrt{(\frac{9}{2})^2 - 4\cdot\frac{9}{2}} = \sqrt{\frac{81}{4}-18} = \sqrt{\frac{81-72}{4}} = \sqrt{\frac{9}{4}} = \frac{3}{2}$

$P = \frac{9}{2}-3 = \frac{9-6}{2} = \frac{3}{2} \qquad L = P$

$\mathcal{K} = \{\frac{9}{2}\}$

b) $\sqrt{n-1} - 2\sqrt{2n-3} = 0$

$\sqrt{n-1} = 2\sqrt{2n-3} \quad /^2$

$n-1 = 4(2n-3)$

$n-1 = 8n-12$

$11 = 7n$

$n = \frac{11}{7}$

Zk.: $L = \sqrt{\frac{11}{7}-1} - 2\sqrt{2\cdot\frac{11}{7}-3} =$

$= \sqrt{\frac{11-7}{7}} - 2\sqrt{\frac{22-21}{7}} =$

$= \sqrt{\frac{4}{7}} - 2\sqrt{\frac{1}{7}} = \frac{2}{\sqrt{7}} - 2\frac{1}{\sqrt{7}} =$

$= \frac{2}{\sqrt{7}} - \frac{2}{\sqrt{7}} = 0 \qquad P = 0 \qquad L = P$

$\mathcal{K} = \{\frac{11}{7}\}$

c) $\sqrt{n+5} - \sqrt{n^2-7} = 0$

$\sqrt{n+5} = \sqrt{n^2-7}$

$n+5 = n^2-7$

$0 = n^2-n-12$

$0 = (n-4)(n+3)$

$n_1 = 4 \qquad n_2 = -3$

Zk. $L(4) = \sqrt{9} - \sqrt{16-7} = 3-3 = 0$

$P(4) = 0 \qquad L(4) = P(4)$

$L(-3) = \sqrt{-3+5} - \sqrt{9-7} = \sqrt{2} - \sqrt{2} = 0$

$P(-3) = 0 \qquad L(-3) = P(-3)$

$\mathcal{K} = \{-3, 4\}$

d) $n + \sqrt{2+n} = 4$

$\sqrt{2+n} = 4-n$

$2+n = 16-8n+n^2$

$0 = n^2-9n+14$

$0 = (n-7)(n-2)$

$n_1 = 7 \qquad n_2 = 2$

Zk. $L(7) = 7 + \sqrt{2+7} = 7+3 = 11$

$P(7) = 11 \qquad L(7) \neq P(7)$

$L(2) = 2 + \sqrt{2+2} = 2+2 = 4$

$P(2) = 4 \qquad L(2) = P(2)$

$\mathcal{K} = \{2\}$

④ Řešte v $\mathbb{R}$

a) $\sqrt{3s+1} = 2+\sqrt{s-1}$ $/^2$

$3s+1 = 4+4\sqrt{s-1}+s-1$

$2s-2 = 4\sqrt{s-1}$ $/:2$

$s-1 = 2\sqrt{s-1}$ $/^2$

$s^2-2s+1 = 4\cdot(s-1)$

$s^2-2s+1 = 4s-4$

$s^2-6s+5=0$

$(s-5)(s-1)=0$

$s_1=5 \quad s_2=1$

Zk.: $L(5)=\sqrt{16}=4$

$P(5)=2+\sqrt{4}=4 \quad L(5)=P(5)$

$L(1)=\sqrt{3+1}=2$

$P(1)=2+\sqrt{0}=2 \quad L(1)=P(1)$

$\mathcal{K}=\{1;5\}$

b) $\sqrt{2x-4} = 1+\sqrt{x+5}$ $/^2$

$2x-4 = 1+2\sqrt{x+5}+x+5$

$x-10 = 2\sqrt{x+5}$ $/^2$

$x^2-20x+100 = 4\cdot(x+5)$

$x^2-20x+100 = 4x+20$

$x^2-24x+80=0$

$(x-20)(x-4)=0$

$x_1=20 \quad x_2=4$

Zk.: $L(20)=\sqrt{36}=6$

$P(20)=1+\sqrt{25}=6 \quad L(20)=P(20)$

$L(4)=\sqrt{4}=2$

$P(4)=1+\sqrt{9}=1+3=4 \quad L(4)\neq P(4)$

$\mathcal{K}=\{20\}$

c) $\sqrt{8+t}+\sqrt{8-t}=4$

$\sqrt{8+t} = 4-\sqrt{8-t}$

$8+t = 16-8\sqrt{8-t}+8-t$

$2t-16 = -8\sqrt{8-t}$ $/:2$

$t-8 = -4\sqrt{8-t}$ $/^2$

$t^2-16t+64 = 16\cdot(8-t)$

$t^2-16t+64 = 128-16t$

$t^2-64=0$

$(t-8)(t+8)=0$ nebo $t^2=64$

$t_1=8 \quad t_2=-8$ $|t|=8$

$t_{1,2}=\pm 8$

$L(8)=\sqrt{16}+\sqrt{0}=4 \quad P(8)=4 \quad L(8)=P(8)$

$L(-8)=\sqrt{0}+\sqrt{8-(-8)}=\sqrt{16}=4$

$P(-8)=4 \quad L(-8)=P(-8)$

$\mathcal{K}=\{-8;8\}$

d) $\sqrt{1-x}-2\sqrt{x}=2$

$\sqrt{1-x} = 2+2\sqrt{x}$ $/^2$

$1-x = 4+8\sqrt{x}+4x$

$-5x-3 = 8\sqrt{x}$ $/^2$

$(-5x-3)^2 = 64x$

$25x^2+30x+9=64x$

$25x^2-34x+9=0$

$(a=25;\ b=-34;\ c=+9)$

$x_{1,2}=\frac{34\pm\sqrt{34^2-4\cdot25\cdot9}}{50}=\frac{34\pm\sqrt{256}}{50}$

$x_{1,2}=\frac{34\pm16}{50}=\begin{cases}\frac{34+16}{50}=1\\ \frac{34-16}{50}=\frac{18}{50}=\frac{9}{25}\end{cases}$

$(-a-b)^2 = (-a-b)(-a-b) = -1(a+b)(-1)(a+b) = (a+b)^2$

$L(1)=\sqrt{0}-2\sqrt{1}=-2$

$P(1)=2 \quad L(1)\neq P(1)$

$L(\frac{9}{25})=\sqrt{1-\frac{9}{25}}-2\sqrt{\frac{9}{25}}=$

$=\sqrt{\frac{16}{25}}-2\cdot\frac{3}{5}=\frac{4}{5}-\frac{6}{5}=-\frac{2}{5}$

$P(\frac{9}{25})=2 \quad L(\frac{9}{25})\neq P(\frac{9}{25})$

$\mathcal{K}=\emptyset$

③ IRACIONÁLNÍ NEROVNICE

- způsobů řešení více
- VÝHODNÉ ŘEŠIT METODOU NULOVÝCH BODŮ
  (jako mocu v souč. a podíl. tvaru – ANULOVAT)

Příklady

① Řešte v R

a) $\sqrt{x^2+2} \geq 2-x$ ANULOVAT

$\sqrt{x^2+2} - 2 + x \geq 0$ (*) $\mathcal{D} = R$ [$x^2+2 \geq 0$ pro $\forall x \in R$]

nul. b. $\sqrt{x^2+2} - 2 + x = 0$ ŘEŠÍME IRAC. RCI, ZKOUŠKA!

$\sqrt{x^2+2} = 2 - x$

$x^2 + 2 = 4 - 4x + x^2$

$4x = 2$

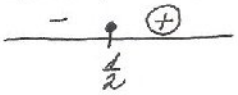
$x = \frac{1}{2}$

$x_0 = \frac{1}{2}$

Zk: $L(\frac{1}{2}) = \sqrt{(\frac{1}{2})^2+2} - 2 + \frac{1}{2} = \sqrt{\frac{1}{4}+2} - \frac{3}{2} = \sqrt{\frac{1+8}{4}} - \frac{3}{2} = \sqrt{\frac{9}{4}} - \frac{3}{2} = \frac{3}{2} - \frac{3}{2} = 0$

$P(\frac{1}{2}) = 0$ $L(\frac{1}{2}) = P(\frac{1}{2})$

nul. b. rozdělí def. obor na intervaly, zjistíme znam. změny v $\mathcal{D}$



dos. do (*)

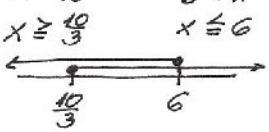
zvol. 1: $\sqrt{1^2+2} - 2 + 1 = \sqrt{3} - 1 > 0$ (⊕)

b) $\sqrt{3x-10} > \sqrt{6-x}$ ANULOVAT

$\sqrt{3x-10} - \sqrt{6-x} > 0$ (*) $\mathcal{D}$: $3x - 10 \geq 0 \wedge 6 - x \geq 0$

$3x \geq 10$ $6 \geq x$

$x \geq \frac{10}{3}$ $x \leq 6$



$\mathcal{D} = \langle \frac{10}{3}, 6 \rangle$

nul. b.

$\sqrt{3x-10} - \sqrt{6-x} = 0$

$\sqrt{3x-10} = \sqrt{6-x}$

$3x - 10 = 6 - x$

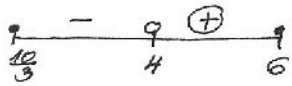
$4x = 16$

$x_0 = 4$

Zk. $L(4) = \sqrt{3 \cdot 4 - 10} - \sqrt{6-4} = \sqrt{2} - \sqrt{2} = 0$

$P(4) = 0$ $L(4) = P(4)$

– znam. změny ZJIŠŤUJEME POUZE V $\mathcal{D}$



dos. do (*)

zvol. 5 (v $\mathcal{D}$)

$\sqrt{3 \cdot 5 - 10} - \sqrt{6-5} = \sqrt{5} - 1 > 0$ ⊕

$\mathcal{K} = (4, 6\rangle$

c) $2\sqrt{x-1} < x$

$2\sqrt{x-1} - x < 0$ (*)

$\mathcal{D}$: $x - 1 > 0$, $x > 1$ $\mathcal{D} = (1, +\infty)$

nul. b. $2\sqrt{x-1} - x = 0$

$2\sqrt{x-1} = x$

$4(x-1) = x^2$!

$4x - 4 = x^2$

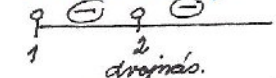
$x^2 - 4x + 4 = 0$

$(x-2)^2 = 0$

$x_{1,2} = 2$ (dvojnás.) nul. b.

Zk: $L = 2\sqrt{2-1} - 2 = 0$ $P = 0$ $L = P$

znam. změny v $\mathcal{D}$



dos. do (*)

zvol. 3: $2\sqrt{3-1} - 3 = 2\sqrt{2} - 3 < 0$

nemění znam.

[zvol. 5: $2\sqrt{5-1} - 5 = 2 \cdot 2 - 4 = -1 < 0$]

$\mathcal{K} = (1, 2) \cup (2, +\infty)$

② Řešte v R

DÚ a) $\sqrt{x^2+2x+5} > x-7$ ANULOVAT

$\sqrt{x^2+2x+5} - x + 7 > 0$

nul. b. $\sqrt{x^2+2x+5} = x-7$

$x^2+2x+5 = x^2-14x+49$

$16x = 44$

$x = \frac{44}{16} = \frac{11}{4}$

Zk: $L = \sqrt{\left(\frac{11}{4}\right)^2 + 2\cdot\frac{11}{4}+5} - \frac{11}{4} + 7 =$

$= \sqrt{\frac{121}{16}+\frac{88}{16}+\frac{80}{16}} - \frac{11}{4} + 7$

$= \sqrt{\frac{289}{16}} - \frac{11}{4} + 7 = \frac{17}{4} - \frac{11}{4} + 7 =$

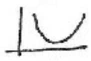
$= \frac{6}{4} + 7 = \frac{3}{2} + 7 = 1\frac{1}{2} + 7 = 8\frac{1}{2}$

$P = 0$ $L \neq P$ $x = \frac{11}{4}$ NENÍ NUL. B. !

D: $x^2+2x+5 \geq 0$

$x_{1,2} = \frac{-2 \pm \sqrt{4-4\cdot5}}{2}$, $D<0$

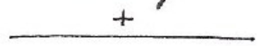
nem. nul. b.



– pro $\forall x \in R$ je $x^2+2x+5 \geq 0$

$D = R$

znam. změny



v D se nemění zn. (není nul. b.)

zv. 0: $\sqrt{5} - 0 + 7 > 0$

$K = D = R$

b) $\sqrt{2+x} < x$ D: $x+2 \geq 0$, $x \geq -2$ $D = \langle -2; +\infty)$

$\sqrt{x+2} - x < 0$ (*)

nul. b.

$\sqrt{x+2} = x$

$x+2 = x^2$

$x^2 - x - 2 = 0$

$(x-2)(x+1) = 0$

$x_1 = 2$ $x_2 = -1$

Zk. $L(2) = \sqrt{4} - 2 = 0$

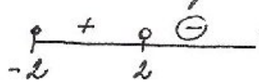
$P(2) = 0$ $L(2) = P(2)$

$L(-1) = \sqrt{1} - (-1) = 2$

$P(-1) = 0$ $L(-1) \neq P(-1)$

tj. 1 NUL. BOD $x = 2$

znam. změny v D



zv. vhodně 14: (*)

$\sqrt{16} - 14$

$= 4 - 14 < 0$

$K = (2; +\infty)$

c) $\sqrt{-2+x} < x$ D: $-2+x \geq 0$, $x \geq 2$ $D = \langle 2; +\infty)$

$\sqrt{x-2} - x < 0$ (*)

nul. b.

$\sqrt{x-2} - x = 0$

$\sqrt{x-2} = x$

$x - 2 = x^2$

$x^2 + x - 2 = 0$

$(x+2)(x-1) = 0$

$x_1 = -2$ $x_2 = 1$

Zk.

$L(-2) = \sqrt{-2-2} - (-2)$, nelze, není kořen

$L(1) = \sqrt{1-2} - 1$, nelze, není kořen

nem. nul. b.

znam. změny v D

zv. vhodně 6: $\sqrt{6-2} - 6$

$= \sqrt{4} - 6 = 2 - 6$

$= -4 < 0$ ⊖

$K = D = \langle 2; +\infty)$